

EXAMPLES OF $B(D, \lambda)$ -REFINABLE AND WEAK $\bar{\theta}$ -REFINABLE SPACES

STEPHEN H. FAST AND J. C. SMITH

ABSTRACT. In 1980, J. C. Smith asked for examples which would demonstrate the relationships between the properties $B(D, \lambda)$ -refinability, $B(D, \omega_0)$ -refinability, and weak $\bar{\theta}$ -refinability. This paper gives such examples in the class of T_4 spaces.

1. INTRODUCTION AND DEFINITIONS

In [5], J. C. Smith asks for an example showing the relationship between $B(D, \lambda)$ -refinability and $B(D, \omega_0)$ -refinability where λ represents a countable ordinal. Smith also conjectured that weak $\bar{\theta}$ -refinability is strictly weaker than $B(D, \omega_0)$ -refinability. This paper gives examples demonstrating the relationship between these properties.

In §2 we construct a T_4 space $K_{\omega_1}^*$ such that for each $\alpha < \omega_1$, $K_{\omega_1}^*$ has a closed subspace that is $B(D, \alpha)$ -refinable but not $B(D, \beta)$ -refinable for any $\beta < \alpha$. We then show that $K_{\omega_0+1}^*$ is weak $\bar{\theta}$ -refinable, implying that weak $\bar{\theta}$ -refinability is strictly weaker than $B(D, \omega_0)$ -refinability.

Definition 1. A space is said to be *mesocompact* if every open cover of the space has an open refinement \mathcal{V} such that any compact subset of the space meets only finitely many members of \mathcal{V} .

Definition 2. A space is said to be *metacompact* if every open cover of the space has a point-finite open refinement.

Definition 3. A space X is said to be $B(D, \lambda)$ -refinable provided that for every open cover $\mathcal{U} = \{V_\delta | \delta \in \Lambda\}$ of X there exists a family $\{\mathcal{B}_\alpha : \alpha \in \lambda\}$ of partial refinements of \mathcal{U} such that the following conditions hold:

- (1) $\bigcup \{\mathcal{B}_\alpha : \alpha \in \lambda\}$ is a refinement of \mathcal{U} ;
- (2) $\bigcup_{\gamma < \alpha} (\bigcup \mathcal{B}_\gamma)$ is closed in X for every $\alpha \in \lambda$;
- (3) \mathcal{B}_α is a relatively discrete closed collection in $X - \bigcup (\bigcup_{i < \alpha} \mathcal{B}_i)$.

In this case $\mathcal{B} = \bigcup \{\mathcal{B}_\alpha | \alpha \in \lambda\}$ is said to be a $B(D, \lambda)$ refinement of \mathcal{U} . Note that if we define for each $\alpha \in \lambda$ the family $\mathcal{H}_\alpha = \{H_\delta | \delta \in \Lambda\}$ where

Received by the editors August 10, 1990 and, in revised form, February 11, 1994; originally communicated to the *Proceedings of the AMS* by D. K. Burke.

1991 *Mathematics Subject Classification*. Primary 54D18, 54D20, 54G20; Secondary 54D15.

Key words and phrases. $B(D, \lambda)$ -refinable, weak $\bar{\theta}$ -refinable, metacompact, mesocompact, θ -refinable, compact, Bing space, collectionwise T_2 , normal, standard subbasic funnel.

$H_\delta = \bigcup \{B \in \mathcal{B}_\alpha \mid B \subseteq U_\delta \text{ and } B \not\subseteq U_\gamma \text{ for } \gamma < \delta\}$, then $\mathcal{H} = \bigcup \{\mathcal{H}_\alpha \mid \alpha \in \lambda\}$ is also a $B(D, \lambda)$ refinement of \mathcal{U} . We say that \mathcal{H} is the *amalgamation* of \mathcal{B} . Thus we may henceforth assume that all $B(D, \lambda)$ refinements are amalgamated. Note that each \mathcal{H}_α is a one-to-one partial refinement of \mathcal{U} .

Definition 4. Let $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ be a collection of subsets of a space X . Then for $x \in X$ we define $\text{ord}(x, \mathcal{H}) = |\alpha \in A : x \in H_\alpha|$.

Definition 5. A space X is said to be θ -refinable provided that for every open cover \mathcal{U} of X there exists a sequence $\{\mathcal{S}_n : n \in \omega_0\}$ of open refinements of \mathcal{U} such that for each $x \in X$ there exists some $n_x \in \omega_0$ such that $0 < \text{ord}(x, \mathcal{S}_{n_x}) < \infty$.

Definition 6. A space X is said to be weak $\bar{\theta}$ -refinable provided that for every open cover \mathcal{U} of X there exists a sequence $\{\mathcal{S}_n : n \in \omega_0\}$ of open partial refinements of \mathcal{U} such that the following conditions hold:

- (1) $\bigcup \{\mathcal{S}_n : n \in \omega_0\}$ is an open refinement of \mathcal{U} ;
- (2) for each $x \in X$ there exists some $n_x \in \omega_0$ such that $0 < \text{ord}(x, \mathcal{S}_{n_x}) < \infty$;
- (3) for each $x \in X$ there exists some $k_x \in \omega_0$ such that $x \notin \bigcup \{\mathcal{S}_n : n \geq k_x\}$.

Note that $\text{mesocompact} \rightarrow \text{metacompact} \rightarrow \theta\text{-refinable} \rightarrow B(D, \omega_0)\text{-refinable} \rightarrow \text{weak } \bar{\theta}\text{-refinable}$.

It is known that the first three implications are not reversible [4]. This paper shows that the last implication is not reversible in T_4 spaces.

2. CONSTRUCTION OF THE SPACE $K_{\omega_1}^*$

In [1], R. H. Bing gave an example of a normal topological space that is not collectionwise T_2 . We call such an example a *Bing space* and give the construction of such a space below.

Let $Q_1 = P(\omega_1) =$ the set of all subsets of ω_1 . Let $\tilde{G}_1 = \{f \in \prod_{q \in Q_1} \{0, 1\} \mid f \in \Pi_{\{\alpha\}}^{-1}(1) \text{ for some positive, finite number of } \alpha \in \omega_1\}$, where $\{0, 1\}$ is the two-point discrete space. Note that $f \in \Pi_{\{\alpha\}}^{-1}(1)$ if and only if $f(\{\alpha\}) = 1$. For $\alpha \in \omega_1$, define f_α by

$$f_\alpha(q) = \begin{cases} 1 & \text{if } \alpha \in q \\ 0 & \text{if } \alpha \notin q \end{cases}.$$

Define $K_1 = \{f_\alpha \mid \alpha \in \omega_1\}$ so that $K_1 \subseteq \tilde{G}_1$. Let $G_1 = K_1 \cup \{f \in \tilde{G}_1 \mid K_1 \nmid f(q) = 1 \text{ for only finitely many } q \in Q_1\}$. Topologize G_1 by adding to the induced Tychonoff product topology all singleton sets $\{g\}$ where $g \in G_1 - K_1$. Let $K_2^* = G_1$ with the topology described above. For each $\alpha < \omega_1$, let $U_\alpha^2 = \pi_{\{\alpha\}}^{-1}(1)$. We call such a set U_α^2 a standard subbasic open funnel. Since $\mathcal{U}^2 = \{U_\alpha^2 \mid \alpha < \omega_1\}$ covers K_2^* , we call \mathcal{U}^2 the standard open cover of K_2^* . Let $Q_2 = P(G_1 - K_1)$. Using Q_2 , construct K_2 and the Bing space G_2 as in the construction of the Bing space G_1 using Q_1 . Let $K_3^* = G_1 \cup G_2$, identifying $G_1 - K_1$ with K_2 by the bijection $\varphi : (G_1 - K_1) \rightarrow K_2$ defined by the following:

$\varphi(f) = g_f \in K_2$, where

$$g_f(q) = \begin{cases} 1 & \text{if } f \in q \\ 0 & \text{if } f \notin q \end{cases} \text{ for all } f \in G_1 - K_1.$$

Topologize K_3^* as follows. If A is an open subset of K_2^* , then $A \cup \{A_f | f \in K_2 \cap A\}$, where each A_f is a set containing f that is open in G_2 , is a basic open set in K_3^* . In addition, any open set $B \subseteq G_2$ is open in K_3^* . For each $\alpha < \omega_1$, let $U_\alpha^3 = U_\alpha^2 \cup \{f | f \in \pi_{\{k\}}^{-1}(1) \text{ for some } k \in K_2 \cup U_\alpha^2\}$. Note that $U^3 = \{U_\alpha^3 | \alpha < \omega_1\}$ is the standard open cover of K_3^* . We now construct by transfinite recursion the space $K_{\omega_1}^*$.

Let $\gamma < \omega_1$.

Case (1). If $\gamma = \beta + 1$, where β is a successor ordinal, construct K_γ^* from K_β^* as K_3^* was constructed from K_2^* above. For each $\alpha < \omega_1$, let $U_\alpha^\gamma = U_\alpha^\beta \cup \{f | f \in \pi_{\{k\}}^{-1}(1) \text{ for some } k \in K_\beta \cap U_\alpha^\beta\}$. Note that $\mathcal{U}^\gamma = \{U_\alpha^\gamma | \alpha < \omega_1\}$ is the standard open cover of K_γ^* .

Case (2). If γ is a limit ordinal, let $K_\gamma^* = \bigcup \{K_\delta^* | \delta < \gamma\}$ with the natural identification of levels and topology.

Case (3). If γ is the successor of a limit ordinal, say $\gamma = \beta + 1$, let $\psi : \omega \rightarrow \beta$ be an increasing cofinal map where each $\psi(n)$ is a successor ordinal. Let $K_\gamma = \{\varphi : \omega \rightarrow K_\beta^* | \varphi(n) \in K_{\psi(n)} \text{ for every } n \in \omega \text{ and a tail of the image of } \varphi \text{ is contained in some member of the standard open cover } \mathcal{U}^\beta \text{ of } K_\beta^*\}$. Define $K_\gamma^* = K_\beta^* \cup K_\gamma$ topologized as follows: If set B is open in K_β^* , then $B \cup \{\varphi \in K_\gamma | \text{a tail of the image of } \varphi \text{ is in } B\}$ is an open set in K_γ^* . In addition, all singleton sets $\{\varphi\}$ for $\varphi \in K_\gamma$ are open.

Define the space $K_{\omega_1}^* = \bigcup \{K_\gamma^* | \gamma < \omega_1\}$ with the natural identification of the levels and topology. Note that $\mathcal{U} = \{U_\delta | \delta < \omega_1\}$ is an open cover of $K_{\omega_1}^*$, which we call the standard open cover, that each K_γ^* is a closed subspace of $K_{\omega_1}^*$, that each K_γ is a relatively closed, discrete subspace of $K_{\omega_1}^* - \bigcup \{K_\beta^* | \beta < \gamma\}$, and that $K_{\omega_1}^*$ is T_4 . To see that $K_{\omega_1}^*$ is normal, suppose that A and B are disjoint nonempty subsets of $K_{\omega_1}^*$. Then $A \cap (K_\gamma \cup K_{\gamma+1})$ and $B \cap (K_\gamma \cup K_{\gamma+1})$ are disjoint nonempty subsets of $K_\gamma \cup K_{\gamma+1}$, which is a normal Bing space, and can be separated by disjoint open sets, say A^* and B^* . Since $A^* \cap K_{\gamma+1}$ and $B^* \cap K_{\gamma+1}$ are disjoint closed subsets of $K_{\gamma+1} \cup K_{\gamma+2}$, a straightforward induction can be used to construct disjoint open subsets of $K_{\omega_1}^*$ separating A and B .

We now state our main result.

Theorem 1. *For every countable ordinal α , there exists a T_4 , $B(D, \alpha)$ -refinable space which is not $B(D, \beta)$ -refinable for any $\beta < \alpha$.*

Proof. Since each K_α is a relatively closed discrete subspace of $K_{\omega_1}^* - (\bigcup_{\gamma < \alpha} K_\gamma | \gamma \text{ is a successor ordinal})$, it is clear that K_α^* is $B(D, \alpha)$ -refinable.

To prove that K_α^* is not $B(D, \beta)$ -refinable for any $\beta < \alpha$, our inductive proof proceeds in the following way. Let $\{\mathcal{B}_\mu | \mu < \alpha\}$ be a collection of partial refinements of \mathcal{U} such that the following conditions hold:

- (1) $\bigcup_{\gamma < \mu} (\bigcup \mathcal{B}_\gamma)$ is closed in $K_{\omega_1}^*$ for every $\mu < \alpha$;

(2) \mathcal{B}_μ is a relatively discrete, closed collection in $K_{\omega_1}^* - (\bigcup_{\gamma < \mu} (\bigcup \mathcal{B}_\gamma))$ for each $\mu < \alpha$.

First, for every limit ordinal $\alpha < \omega_1$ and $x \in (U_\delta \cap K_{\alpha+1}) - \bigcup_{\mu < \alpha} (\bigcup \mathcal{B}_\mu)$, let $S_{(x, \delta)}$ be an open funnel about x such that

$$S_{(x, \delta)} \subseteq U_\delta - \left(\left(\bigcup_{\mu < \alpha} K_\mu \right) \cup \left(\bigcup_{\mu < \alpha} (\bigcup \mathcal{B}_\mu) \right) \right).$$

For every $\alpha < \omega_1$, we will show that there exists a $\delta_\alpha < \omega_1$ such that for every $\delta \geq \delta_\alpha$ there exists a $x(\alpha, \delta) \in K_{\alpha+1} \cap U_\delta$ and an open set $V(\alpha, \delta)$ about $x(\alpha, \delta)$ such that the following conditions hold:

- (1) $\gamma < \alpha \Rightarrow \delta_\gamma < \delta_\alpha$;
- (2) $\gamma < \alpha$ and $\delta \geq \delta_\alpha \Rightarrow V(\alpha, \delta) \subseteq V(\gamma, \delta)$;
- (3) $\delta \geq \delta_\alpha \Rightarrow x(\alpha, \delta) \notin (\bigcup_{\sigma < \delta} U_\sigma) \cup (\bigcup_{\mu < \alpha} (\bigcup \mathcal{B}_\mu))$;
- (4) $\delta \geq \delta_\alpha \Rightarrow V(\alpha, \delta) \cap (\bigcup_{\mu < \alpha} (\bigcup \mathcal{B}_\mu)) = \emptyset$;
- (5) For every limit ordinal $\tau > \alpha$, $\delta \geq \delta_\alpha$ and $x \in K_{\tau+1} \cap V(\alpha, \delta) \Rightarrow S_{(x, \delta)} \subseteq V(\alpha, \delta)$.

For $\alpha = 0$, let $\delta_\alpha = 0$ and $x(\alpha, \delta) = f_\delta$ for every $\delta < \omega_1$. Let $V(\alpha, \delta) = U_\delta$ for each $\delta < \omega_1$. Then conditions (1)–(5) are satisfied for $\alpha = 0$.

Suppose that $\alpha > 0$ and that conditions (1)–(5) above hold for every $\beta < \alpha$. We show that these conditions hold for α .

I. Case 1. α is a limit ordinal.

Since $\alpha < \omega_1$, choose $\psi < \omega_1$ such that $\psi > \delta_\beta$ for every $\beta < \alpha$. Hence for every $\delta > \psi$, there exists for each $\beta < \alpha$, a $x(\beta, \delta) \in K_{\beta+1} \cap U_\delta$ and an open set $V(\beta, \delta)$ about $x(\beta, \delta)$ such that the inductive conditions (1)–(5) hold. Let $\delta_\alpha = \psi$.

For each $\delta > \psi$, let $x(\alpha, \delta) = \varphi_\delta \in K_{\alpha+1}$, where $\varphi_\delta(\beta) = x(\beta, \delta)$ for every β in the image of ψ_α (see above the construction Case 3 of $K_{\gamma+1}$, where γ is a limit ordinal). By inductive conditions (2) and (3), each $x(\alpha, \delta)$ is well defined and $x(\alpha, \delta) \notin (\bigcup_{\sigma < \delta} U_\sigma) \cup (\bigcup_{\mu < \alpha} (\bigcup \mathcal{B}_\mu))$. By the choice of ψ it follows that $\gamma < \alpha \Rightarrow \delta_\gamma < \psi$. Since if $\delta \geq \psi$, we have $x(\alpha, \delta) \notin \bigcup_{\mu < \alpha} (\bigcup \mathcal{B}_\mu)$, by the definition of the sets $V(\gamma, \delta)$, we can choose an open set $V(\alpha, \delta)$ about $x(\alpha, \delta)$ such that (2), (4), and (5) hold.

Before we consider the case where α is not a limit ordinal, we need the lemma below.

Lemma 1. Let \mathcal{U} be the standard subbasic open cover of $K_{\omega_1}^*$. Fix $\beta < \omega_1$ where β is a successor ordinal. Suppose that X is a closed subset of $K_{\omega_1}^*$ and $\mathcal{B} = \{B_{f_\alpha} | f_\alpha \in K_1\}$ is a relatively discrete closed collection in $K_{\omega_1}^* \setminus X$ which partially refines \mathcal{U} such that the following conditions hold:

- (A) There exists an $\alpha_1 \in \omega_1$ such that for every $\gamma > \alpha_1$ there exists some
- (1) $g(\beta, \gamma) \in U_\gamma \cap K_\beta \setminus (X \cup (\bigcup \{U_\tau | \tau < \gamma\}))$

and

- (2) for each $g(\beta, \gamma)$ in (1) we can choose a funnel $V(\beta, \gamma) \subseteq U_\gamma$ about $g(\beta, \gamma)$ such that $V(\beta, \gamma) \cap X = \emptyset$ and $V(\beta, \gamma)$ hits at most one member of \mathcal{B} ; i.e. $V(\beta, \gamma) \cap B \neq \emptyset$ iff $g(\beta, \gamma) \in B$ for each $B \in \mathcal{B}$. (*)

- (B) Then there exists an $\alpha_2 \in \omega_1$ such that for every $\rho > \alpha_2$ we have

$$\left[V(\beta, \rho) \setminus \left(\left(\bigcup \mathcal{B} \right) \cup \left(\bigcup \{U_\tau | \tau < \rho\} \right) \right) \right] \cap K_{\beta+1} \neq \emptyset.$$

Remark. Note that in (A), if (1) holds, then (2) follows.

Proof. Assume (A) and suppose (B) is false; that is, no such α_2 exists. Choose $\gamma_0 > \alpha_1$ and a funnel $V(\beta, \gamma_0)$ such that

$$[V(\beta, \gamma_0) \setminus ((\bigcup \mathcal{B}) \cup (\{U_{f_i} | \tau < \gamma_0\}))] \cap K_{\beta+1} = \emptyset.$$

By our supposition we can choose $\gamma_1 > \gamma_0$ such that $\gamma_1 > \tau$ if $V(\beta, \gamma_0) \cap B_{f_i} \neq \emptyset$ and

$$[V(\beta, \gamma_1) \setminus ((\bigcup \mathcal{B}) \cup (\{U_{f_i} | \tau < \gamma_1\}))] \cap K_{\beta+1} = \emptyset.$$

Assume that for $\rho < \Gamma$ γ_ρ has been chosen such that the following conditions hold:

- (i) $\gamma_\rho > \gamma_\delta$ if $\delta < \rho$;
- (ii) $\gamma_\rho > \tau$ if $V(\beta, \gamma_\delta) \cap B_{f_i} \neq \emptyset$ for any $\delta < \rho$;
- (iii) $[V(\beta, \gamma_\rho) \setminus ((\bigcup \mathcal{B}) \cup (\{U_{f_i} | \tau < \gamma_\rho\}))] \cap K_{\beta+1} = \emptyset$.

By our supposition there exists

$$\gamma_\Gamma > \sup(\{\gamma_\rho | \rho < \Gamma\} \cup \{\tau \in \omega_1 | B_{f_i} \cap V(\beta, \gamma_\rho) = \emptyset \text{ for some } \rho < \Gamma\})$$

such that (i), (ii), and (iii) hold. Thus we can continue the induction on ω_1 .

Since the singletons $\{g(\beta, \gamma_\delta) | \delta \in \omega_1\}$ cannot be separated by pairwise disjoint open sets in $G_\beta = K_\beta \cup K_{\beta+1}$, and since $V(\beta, \gamma_\delta) \cap G_\beta$ is open in G_β for every $\delta \in \omega_1$, there exists $\delta_1, \delta_2 \in \omega_1$ ($\delta_1 < \delta_2$), such that $V(\beta, \gamma_{\delta_1}) \cap V(\beta, \gamma_{\delta_2}) \cap K_{\beta+1} \neq \emptyset$. Now $G_\beta \cap V(\beta, \gamma_{\delta_1}) \cap V(\beta, \gamma_{\delta_2}) = \bigcap_{j=1}^k \Pi_{q_k}^{-1}(t_k)$ for some $q_1, q_2, \dots, q_k \in Q_\beta$, and each t_j has the value 1 or 0. Since $g(\beta, \gamma_{\delta_1}) \in V(\beta, \gamma_{\delta_1})$ and $g(\beta, \gamma_{\delta_2}) \in V(\beta, \gamma_{\delta_2})$, by (A) if any $q_j = \{h\}$, where $h \in \bigcup \{U_{f_i} | \tau < \gamma_{\delta_1}\} \cap K_\beta$, it follows that $t_j = 0$. Hence $[V(\beta, \gamma_{\delta_1}) \cap V(\beta, \gamma_{\delta_2}) \setminus (\bigcup \{U_{f_i} | \tau < \gamma_{\delta_1}\})] \cap K_{\beta+1} \neq \emptyset$. Choose $x \in [V(\beta, \gamma_{\delta_1}) \cap V(\beta, \gamma_{\delta_2}) \setminus (\bigcup \{U_{f_i} | \tau < \gamma_{\delta_1}\})] \cap K_{\beta+1}$. By (iii) above, we must have that $x \in \bigcup \mathcal{B}$. Choose $B_x \in \mathcal{B}$ with $x \in B_x$. Then

$$(**) \quad x \in V(\beta, \gamma_{\delta_1}) \cap V(\beta, \gamma_{\delta_2}) \cap B_x.$$

By (A-2) above, $g(\beta, \gamma_{\delta_1}) \in B_x$. Thus it follows from (ii) and the assumption (A) that $g(\beta, \gamma_{\delta_2}) \notin B_x$. However, by (A-2) we have that $V(\beta, \gamma_{\delta_2}) \cap B_x = \emptyset$, contradicting (**). Therefore, the lemma is proved.

We now consider the case where α is a successor ordinal.

II. Case 2. $\alpha = \beta + 1$.

Since β satisfies the inductive hypothesis (1)–(5), condition (A) of Lemma 1 is satisfied. To see this, let $X = \bigcup_{\mu < \alpha} (\bigcup \beta_\mu)$ and let $\mathcal{B} = \mathcal{B}_\alpha$. Then the α_1 of Lemma 1, condition (A) is fulfilled by δ_β , since for each $\delta \geq \delta_\beta$ the element $x(\beta, \delta) \in (U_{f_\delta} \cap K_\alpha) \setminus ((\bigcup_{\sigma < \delta} U_\sigma) \cup (\bigcup_{\mu < \alpha} (\bigcup \beta_\mu)))$ by inductive hypothesis (3).

Hence by Lemma 1, there exists $\delta_\alpha < \omega_1$ satisfying condition (3). Clearly, we can choose $\delta_\alpha > \delta_\beta$, so (1) is satisfied. Since \mathcal{B}_α is a relatively closed, discrete collection in $K_{\omega_1}^* - (\bigcup_{\mu < \alpha} (\bigcup \beta_\mu))$, we can choose an open set $V(\alpha, \delta)$ about each $x(\alpha, \delta)$ such that conditions (2), (4), and (5) are satisfied. The proof of Theorem 1 is now complete.

Theorem 2. G_n is metacompact for each $n \in N$.

Proof. Let \mathcal{U} be any open cover of G_n . For each $f \in K_n$, choose a member U_f of \mathcal{U} that contains f . For each $f \in K_n$, let V_f represent the standard

funnel about f . Then $\{U_f \cap V_f | f \in K_1\} \cup \{\{g\} | g \in G_n \setminus K_n\}$ is a point-finite refinement of \mathcal{U} .

Theorem 3. $K_{\omega_0}^*$ is metacompact.

Proof. Let $\mathcal{V} = \{V_\alpha | \alpha \in A\}$ be an open cover of $K_{\omega_0}^*$. Then, since G_1 is metacompact, \mathcal{V} has a 1-1, open in $K_{\omega_0}^*$, partial refinement $\mathcal{T} = \{T_\alpha | \alpha \in A\}$ which is point-finite on G_1 and covers G_1 . For each $x \in G_n \cap T_\alpha$, let $C(x, n, \alpha)$ be the intersection of the standard funnel about x with T_α . Define the open set S_α^1 inductively as follows:

- (1) $S_\alpha^1 \cap G_1 = T_\alpha \cap G_1$;
- (2) for $n > 1$, $S_\alpha^1 \cap G_n = (S_\alpha^1 \cap K_n) \cup \{y \in G_n | y \in C(x, n, \alpha) \text{ for some } x \in S_\alpha^1 \cap K_n\}$.

Note that $\mathcal{S}^1 = \{S_\alpha^1 | \alpha \in A\}$ is an open in $K_{\omega_0}^*$, point-finite, partial refinement of \mathcal{V} which covers G_1 . To see this, suppose $x \in K_3$. Then $x \in \Pi_q^{-1}(1)$ for only finitely many q in $P(K_2)$. Hence x is a member of only finitely many standard funnels about elements in K_2 . Since $K_2 \subseteq G_1$, \mathcal{S}^1 is point-finite on K_2 . Thus, by (2) it must be the case that \mathcal{S}^1 is point-finite on K_3 . Continuing in this way it follows that \mathcal{S}^1 is point-finite on K_n for every n .

Next, since $K_1 \cup K_2 \cup \dots \cup K_{n-1}$ is a closed subset of $K_{\omega_0}^*$, and since G_n is metacompact for each n , we can construct a 1-1 point-finite open partial refinement \mathcal{S}^n of \mathcal{V} that covers K_n and misses $K_1 \cup K_2 \cup \dots \cup K_{n-1}$. It then follows that $\mathcal{S} = \bigcup \{\mathcal{S}^n : n \in N\}$ is a point-finite open refinement of \mathcal{V} . Hence, $K_{\omega_0}^*$ is metacompact.

To show that $K_{\omega_0}^*$ is mesocompact, it suffices to show that every compact subset of $K_{\omega_0}^*$ is finite. In [2], J. R. Boone shows that every compact subset of G_1 is finite. We now extend this result to obtain the following lemma.

Lemma 2. For each $n \in N$, if C is a compact subset of $K_{\omega_0}^*$, then $C \cap (\bigcup \{G_i | i \leq n\})$ is finite.

Proof. The proof is by induction on N . For $n = 1$ observe that $G_1 \cap C$ is closed in $K_{\omega_0}^*$ and therefore compact. Suppose that we can choose distinct elements $f_1, f_2, f_3, \dots, f_n, \dots$ in $C \cap G_1$. Since K_1 is discrete, we may assume that each f_i belongs to $K_2 \cap C$. Since $|\{q \in Q_1 | f_i(q) \neq 0 \text{ for some } i \in N\}| \leq \aleph_0$, for each $f \in K_1$ we can choose a basic open funnel V_f about f that misses $\{f_1, f_2, f_3, \dots\}$, since $|\{q \in Q_1 | f(q) \neq 0\}| > \aleph_0$. Thus $\{V_f | f \in K_1 \cap C\} \cup \{\{f\} : f \in C \cap K_2\}$ is an open cover (open in G_1) of $C \cap G_1$ with no finite subcover, contradicting the compactness of $C \cap G_1$. Hence $C \cap G_1$ must be finite.

Assume that for all $k < n$, $C \cap [\bigcup \{G_i : i \leq k\}]$ is finite.

By inductive hypothesis, $C \cap (\bigcup \{G_i | i \leq n-1\})$ is compact and therefore finite. Suppose we can choose distinct elements f_1, f_2, f_3, \dots in $C \cap K_{n+1}$. Since $|\{q \in Q_n | f_i(q) \neq 0 \text{ for some } i \in N\}| \leq \aleph_0$, for each $f \in C \cap (\bigcup \{G_i | i \leq n-1\})$ we can choose a funnel V_f about f which misses $\{f_1, f_2, f_3, \dots\} \subseteq C$. Thus $\{V_f | f \in C \cap (\bigcup \{G_i | i \leq n-1\})\} \cup \{\{f\} : f \in C \cap K_{n+1}\}$ is an open cover of $C \cap (\bigcup \{G_i | i \leq n\})$ with no finite subcover, in contradiction to the compactness of $C \cap (\bigcup \{G_i | i \leq n\})$, so the lemma follows.

Lemma 3. If C is a compact subset of $K_{\omega_0}^*$, then C is finite.

Proof. Suppose C is an infinite compact subset of $K_{\omega_0}^*$. Then by Lemma 2

there must exist a sequence $f : N \rightarrow C$ such that $f(n) \notin \bigcup \{K_j | f(i) \in K_j \text{ for some } i < n\}$. Since $\{f(n) | n \in N\}$ is a closed subset of C , it is compact. It is easy to show that this sequence is also discrete and thus cannot be compact. Hence the lemma is proved.

From Theorem 3 and Lemma 3 above we now have the following.

Theorem 4. $K_{\omega_0}^*$ is mesocompact.

Theorem 5. (i) $K_{\omega_0}^*$ is $B(D, \omega_0)$ -refinable but not $B(D, n)$ -refinable for any n .

(ii) $K_{\omega_0+1}^*$ is $B(D, \omega_0 + 1)$ -refinable but not $B(D, \omega_0)$ -refinable.

(iii) $K_{\omega_0+1}^*$ is weak $\bar{\theta}$ -refinable and T_4 .

Proof. The proof of (i) and (ii) follow directly from Theorem 1. Let \mathcal{U} be an open cover of $K_{\omega_0+1}^*$. Since $K_{\omega_0}^*$ is metacompact, \mathcal{U} has a partial refinement \mathcal{S}_1 , which is point-finite on $K_{\omega_0}^*$. Next, let \mathcal{S}_2 be the collection of singleton subsets of $K_{\omega_0+1}^* \setminus K_{\omega_0}^*$. Then $\mathcal{S}_1 \cup \mathcal{S}_2$ is a weak $\bar{\theta}$ -refinement of \mathcal{U} .

Question. If we modify the construction of each G_n so that

$$G_n = \left\{ f \in \prod_{q \in Q_n} \{0, 1\} \mid f \in \Pi_{\{\alpha\}}^{-1}(1) \text{ for some } \alpha \in \omega_1 \right\},$$

would the space $K_{\omega_0+1}^*$ remain weak $\bar{\theta}$ -refinable? The authors conjecture that it would not. If not, then $K_{\omega_0+1}^*$ is a T_4 , $B(D, \omega_0 + 1)$ -refinable space that is not weak $\bar{\theta}$ -refinable.

ACKNOWLEDGMENT

The authors would like to thank the referees for several helpful comments concerning the paper.

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DEPARTMENT OF MATHEMATICS, BLUEFIELD COLLEGE, BLUEFIELD, VIRGINIA 24605-1799

DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061-0123